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TRANSIENT HEAT EQUATION WITH NATURAL BOUNDARY CONDITIONS – FINITE ELEMENT ANALYSIS

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1. Introduction

Thermal treatment of the building is still current topic of the civil engineering. The temperature field stipulation in various kinds of climatic conditions and various thermal conditions inside even before the building is constructed, is crucial for prediction of thermal insulation efficiency, also other observations are based on it, e.g. dampness in dwelling prevention, etc. There are a lot of publications of civil engineers devoted to this topic as the theoretical ones, so those including the measurement and validations of the results. The governing equation of this phenomenon of heat conduction – heat equation is also very well known. The finite element analysis of heat equation is very well elaborated for essential or essential and natural boundary condition. The paper deals with the finite element analysis of the transient heat equation in three dimensions with natural boundary conditions.

2. Domain Ω : three dimensional thermal bridge

Thermal bridges – the places where the temperature field is deformed – the places which the heat passes through faster than nearby [1], is the crucial conception in building physics. Here, the temperature field deformation is caused by the geometry change or by composition of the wall or by both.

The subject of the investigation presented is a three dimensional fragment of a building construction – joint of the building envelope, internal wall separating two neighbouring rooms and floor/ceiling), see Fig. 1, (the domain Ω). The cutting planes in all three directions are placed in the distance sufficiently far from the thermal bridge. Here ”sufficiently far” means zero flux through them in the direction of outer normal. On the Fig. 1a) there is the composition of the domain sketched. The material properties of particular components of the domain Ω are taken from [2], see Table 1.

The boundary $\partial\Omega$ involves the three cutting areas, symmetry planes (symmetry with regarding to the temperature field mentioned), (indexed R), the wall surfaces exposed to internal surroundings (indexed I), and one wall surface exposed to exterior (indexed E), $I \cup E \cup R = \partial\Omega$, see Fig.1. b).

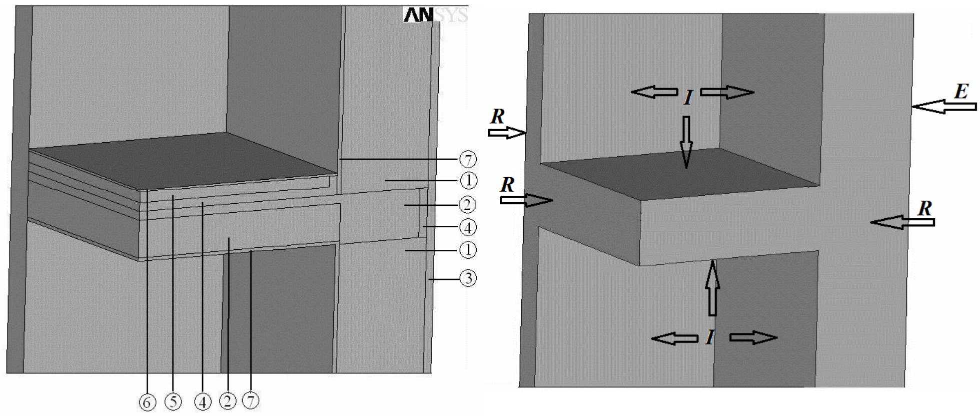

Fig. 1a) Domain composition, b) domain boundary: *I* – interior surfaces, *E* – exterior exposed surfaces, *R* – cutting planes

Table 1 Material properties

No.	Material	heat conductivity λ [W(m K)$^{-1}$]	specific heat c [J(kg K)$^{-1}$]	density ρ [kg/m^3]
1	Porotherm brick	0.2	840	550
2	reinforced concrete	1.58	1020	2400
3	perlite plaster	0.18	850	500
4	extruded polystyrene	0.034	2060	30
5	nivelation layer	1.02	860	2000
6	laminate floor	0.19	1880	1600
7	plaster	0.7	840	1600

3. Governing equation, boundary and initial conditions

Transient heat transfer in solid materials (conduction) is governed by 2nd order partial differential equation, heat equation (1).

$$\rho .c.\frac{\partial T}{\partial t} = -\nabla \cdot (\lambda \nabla T) \quad in\ \Omega \subset R^3. \tag{1}$$

The convection of heat between the bulk air and the domain is described by the *Newton boundary conditions* (2) respectively (3) on the internal / external surfaces *I* respectively *E,* zero fluxes in the outer normal direction through cutting surfaces are represented by homogenous *Neumann boundary condition* (4), the initial temperature distribution in the entire domain Ω is represented by the *initial condition* (5).

$$\lambda \frac{\partial T}{\partial \vec{n}_I} = -\vec{q} \cdot \vec{n}_I = h_I (T - T_I) \quad on\ I \subset \partial\Omega\ , \tag{2}$$

$$-\lambda \frac{\partial T}{\partial \vec{n}_E} = \vec{q} \cdot \vec{n}_E = h_E (T - T_E) \quad on\ E \subset \partial\Omega, \tag{3}$$

$$\frac{\partial T}{\partial \vec{n}_R} = 0 \quad on\ R \subset \partial\Omega, \tag{4}$$

$$T(X,0) = T_0(X) \quad in\ \Omega, \tag{5}$$

Here the unknown function $T = T(X,t) = T(x,y,z,t)$ is the function of three space variables and time, the materials are homogenous and isotropic, the dependence of its physical properties on the temperature (supposed range of temperature) is negligible, i.e. each material property is represented by the unique value.

4. Weak form of the initial-boundary problem

Firstly we get the variation formulation of (1) – by multiplying (1) with weight function w and integrate it on the entire domain Ω [3]:

$$\int_\Omega \left(\rho c \frac{\partial T}{\partial t} \right) w\ d\Omega - \int_\Omega \left(\nabla \cdot (\lambda \nabla T) \right) w\ d\Omega = 0 \quad in\ \Omega \subset R^3. \tag{6}$$

Further we would like to incorporate the boundary conditions to the weak form; so applying the divergence theorem, e.g. [4], to the integrand of the second integral we get

$$\int_\Omega \left(\rho c \frac{\partial T}{\partial t} \right) w\ d\Omega - \int_\Omega \nabla \cdot \left(w(\lambda \nabla T) \right) d\Omega + \int_\Omega \nabla w \cdot (\lambda \nabla T) d\Omega = 0. \tag{7}$$

Let us take the second integral in (7) and rearrange it by using Green formula and consequently write out the particular natural conditions (2) – (4) per each particular surface of $\partial\Omega$, (8). Afterwards let us incorporate (8) to (7) and partitioning the integrands of surfaces integrals

$$\int_\Omega \nabla \cdot \left(w(\lambda \nabla T) \right) d\Omega = \int_{\partial\Omega} w(\lambda \nabla T) \cdot \vec{n}\ dS = \int_E w\, h_E (T - T_E)\ dS + \int_I w\left(-h_I (T - T_I) \right) dS, \tag{8}$$

$$\begin{aligned} &\int_\Omega \left(\rho c \frac{\partial T}{\partial t} \right) w\ d\Omega - \int_E w\, \lambda h_E T\ dS + \int_I w \lambda h_I\ T\ dS + \int_\Omega \nabla w \cdot (\lambda \nabla T) d\Omega = \\ &= -\int_E w\, \lambda h_E T_E\ dS + \int_I w\, \lambda h_I T_I\ dS. \end{aligned} \tag{9}$$

5. Discretization of geometry, discrete representation for unknown variable

At the beginning of the finite element method setting on we discretize the domain Ω to the finite number of subdomains – elements, see Fig. 2, and use the approximation solution (10) instead of an exact one with interpolating functions $T_i(t)$ such that $T_i(t) = T(X_i, t)$ and the shape functions φ [4]. Thereby the continuous physical problem is converted to discrete one with unknown nodal values.

$$T(X,t) = \sum_{i=1}^{n} T_i(t)\varphi_i . \tag{10}$$

By incorporating the approximation solution (10) instead of function T into (9) we get the system of ordinary differential equations with initial conditions following from (5) evaluated for the particular nodes of the finite element net, (11), i.e. (12) in the matrix form [5].

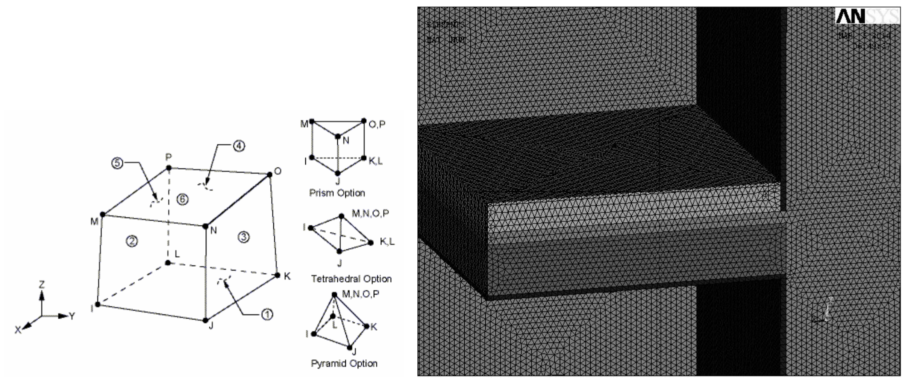

Fig. 2a) Element type, b) domain Ω discretized – meshed

$$\sum_i \int_\Omega \rho c \varphi_i \varphi_j d\Omega \frac{\partial T_i(t)}{\partial t} + \lambda \left[\sum_i \int_\Omega \nabla \varphi_i \cdot \nabla \varphi_j d\Omega - h_E \int_E \varphi_i \varphi_j \, dS + h_I \int_I \varphi_i \varphi_j dS \right] T_i(t) =$$
$$= -\int_E w \lambda h_E T_E \, dS + \int_I w \lambda h_I T_I \, dS \ , \tag{11}$$

$$\begin{aligned} &\mathbf{M}\frac{dT}{dt} + \mathbf{K}T = F, \\ &T_i(0) = T_0(X_i) \quad i = 1,...m. \end{aligned} \tag{12}$$

6. Implementation in FEM software, results

The discretization of the domain, see Fig 2b), is realized by using the “brick” (six walls 8 nodes, or optional shapes, Fig. 2a)) element type with temperature as the primary variable given. After the meshing is ready (over 3 millions elements) and the material properties are inputted and the boundary and initial condition are stipulated, the solution can be executed. As expected, the resulting temperature field (equilibrium state performed), see Fig. 3, reveals the critical points (in the corners of upper and lower rooms). The assumption is underlined by the 2D sections of the 3D temperature field, see Fig. 4. As approaching with the section toward the thermal bridge and tracing the temperature on the internal surface along the floor and the envelope intersecting edge, the temperature decreases towards the thermal bridge.

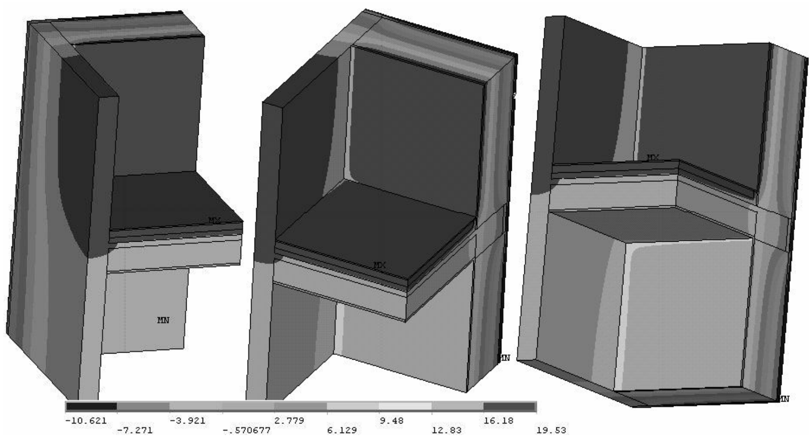

Fig. 3 Temperature field in the domain in equilibrium state, 3 views

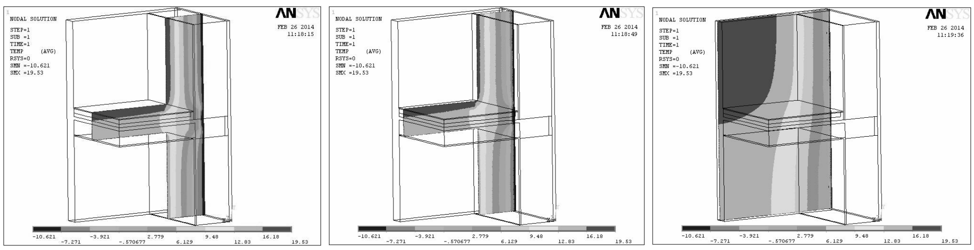

Fig. 4 2D sections of the 3D temperature field in various distances from the thermal bridge

7. Conclusion

Thermal bridges are inspected by civil engineers during the building is being designed or a retrofitting of an older building is planned. Obviously it is carried out in the steady state, in 2D. The aim is the optimization of the construction composition for the sake of the best possible thermal properties, minimization of the heat looses, avoiding the problems related to the high relative humidity on the cold places, etc. Sometimes it is necessary to observe the time dependent temperature field, moreover in 3D.

Thermal performance of building is mostly based on the temperature fields stipulation. The most obvious boundary condition that corresponds to the usual physical situations is convection on the surfaces, Newton boundary condition.

Denotations of symbols

FEM – finite element method, t – time [s], ρ – density [kg/m^3], λ – heat conduction coefficient [W(mK)$^{-1}$], c – specific heat [J (kg K)$^{-1}$], T – temperature [K], T_E, T_I – temperature of the external/internal air [K], $T_0(X)$ – space function of initial temperature field in the domain [K], q – heat flow rate [W/m^2], h_E, h_I – film coefficient (E external, I internal) [W/(m^2.K)], $\vec{n}, \vec{n}_E, \vec{n}_I, \vec{n}_R$ – outer normal vector to a plane (plane name in index), w – weighted function, φ – basis function, Ω – domain, $\partial\Omega$ – boundary of the domain Ω, m – number of the nodes in the domain discretization, **K –** stiffness matrix**, M –** damping matrix

References

[1] Halahyja M., Chmúrny I., Sternová Z.: Thermal Engineering of Buildings. Thermal Protection of Buildings. Jaga Group, 1998

[2] STN 73 05 40-3 Thermal Properties of Building Structures and Buildings. Thermal Protection of the Buildings, Part 3: The properties of the environment and building components

[3] Michlin S. G.: Variational Methods in Mathematical Physics, ALFA Bratislava 1975, ISBN 63-063-75

[4] Rektorys,K.: The Method of Discretization in Time and Partial Differential Equations, Prague 1982

[5] http://de.wikipedia.org/wiki/Linienmethode

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NESTACIONÁRNA ROVNICA VEDENIA TEPLA S PRIRODZENÝMI OKRAJOVÝMI PODMIENKAMI – KONEČNOPRVKOVÁ ANALÝZA.

Resumé

Príspevok popisuje a rozpracuváva matemetický základ pre metódu konečných prvkov pre riešenie počiatočno–okrajového problému tvoreného nestacionárnou rovnicou vedenia tepla a Newtonovými a Neumannovými okrajovými podmienkami. Popisuje tvorbu slabej formulácie prislúchajúcej k tejto úlohe, konkretizuje zavedenie aproximačného riešenia a tvorbu systému obyčajných diferenciálnych rovníc s časovou deriváciou pomocou metódy priamok [5]. V poslednej kapitole je uvedený príklad výsledkov počítačovej implementácie v konečnoprvkovom programe.