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BUCKLING OF VARIABLE CROSS-SECTIONS COLUMNS IN THE BRACED AND SWAY PLANE FRAMES

Ružica R. NIKOLIĆ[1), 2)], Jelena M. DJOKOVIĆ[3)], Ján BUJŇÁK[4), 5)]
[1)] Faculty of Engineering, University of Kragujevac, Kragujevac, Serbia
[2)] Research Center, University of Žilina, Žilina, Slovakia
[3)] Technical Faculty in Bor, University of Belgrade, Bor, Serbia
[4)] Faculty of Civil Engineering, University of Žilina, Žilina, Slovakia
[5)] Faculty of Civil Engineering and Architecture, Opole University of Technology, Opole, Poland

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1. Introduction

Steel parts of the variable cross-section are usually used as columns in various steel structures, like the buildings' frames (skeletons), crane bridges and masts, i.e. always where it is needed to reduce the weight of the structure itself and consecutively lower the manufacturing costs. Numerous researchers were dealing with analysis of the columns with variable cross-sections stability. Li [1] has analyzed the columns with variable cross-sections on elastic supports for the cases of the concentrated and distributed load. He also presented the special solution for the bending moment in lateral buckling for eight cases of different beams with variable cross-sections in [2]. Ermopoulos [3] has considered the problem of buckling of columns with variable cross-sections subjected to stepped axial loading. Raftoyianis and Ermopoulos [4] have studied the elastic stability of eccentrically loaded column with variable cross-sections, with existing initial imperfections. Sing and Li have analyzed in [5] the buckling of columns with variable geometrical and material characteristics, i.e. elastically restrained non-uniform columns made of the functionally graded materials. Li, Xi and Huang have conducted optimization of the composite columns, as well as their stability analysis, shown in [6]. Ermopoulos has presented in [7] the nonlinear equilibrium equations for the non-uniform frame members. By solving these equations, he obtained the ratio of the effective buckling lengths of the considered members and their corresponding critical buckling forces.

The frames with the fixed nodes are currently classified as **braced** ones [8]. They must possess a bracing system which is adequately stiff. When bracing is provided, it is normally used to prevent, or at least to restrict, sway in frames. Common bracing systems are trusses or shear walls. Such bracing systems should resist all the horizontal loads applied to the frames they are bracing.

For frames without a bracing system, and also for frames with a bracing system, but which is not sufficiently stiff to allow classification of the frame as braced, the structure is classified as **unbraced** - with the movable nodes. In all the cases of unbraced frames, a single structural system, consisting of the frame and of the bracing when present, should be analyzed for both the vertical and horizontal loads, acting together, as well as for the effects of imperfections. The existence of a bracing system in a structure does not guarantee that the frame is to be classified as braced. Only when the bracing system reduces the horizontal displacements by at least 80%, the frame can be classified as braced.

Also a ***non-sway*** frame can be classified with the fixed nodes when its response to in-plane horizontal forces is sufficiently stiff for it to be acceptable to neglect any additional forces or moments arising from horizontal displacements of its nodes. Normally, a frame with bracing is likely to be classified as non-sway while an unbraced frame is likely to be classified as ***sway*** one. However, it is important to note that it is theoretically possible for an unbraced frame to be classified as a non-sway, while a frame with bracing may be classified as sway. When a frame is classified as non-sway, a first-order analysis may always be used. The classification of a frame structure as sway or non-sway is based on the value of the ratio of the design value of the total vertical design load V_{Ed} applied to the structure to its elastic critical value V_{cr} producing sway instability. When this ratio is $V_{Ed} / V_{cr} \leq 0.1$, the structure is non-sway.

2. Problem formulation

The critical buckling load and effective buckling lengths, for the rectangular plane frame with columns of the variable cross-sections in Fig. 1, is the subject of investigation in this paper while the deformation modes of this frame are shown in Fig. 2. The first of the four possible deformation modes, shown in Fig. 2, is the case of the frame with the movable nodes, while the other three are frames with the fixed nodes.

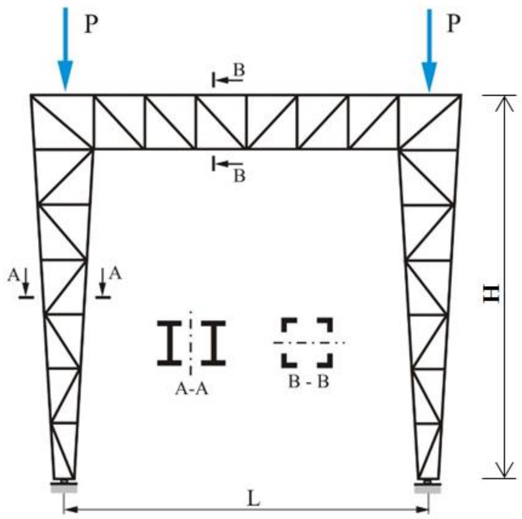


Fig. 1. Geometry of the rectangular plane frame.

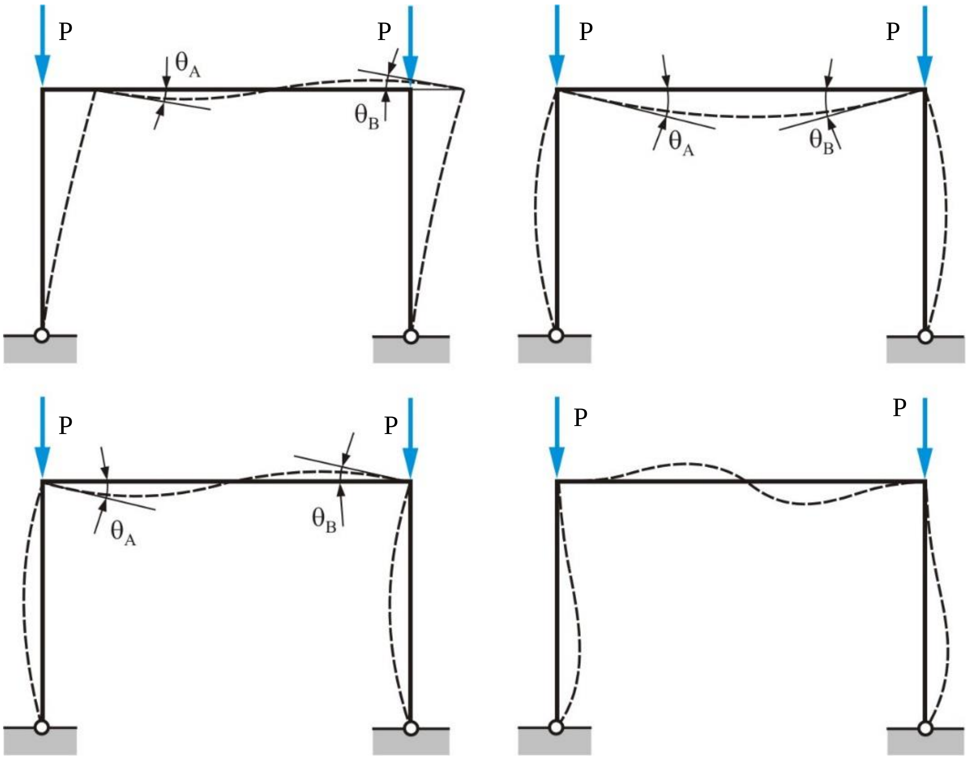


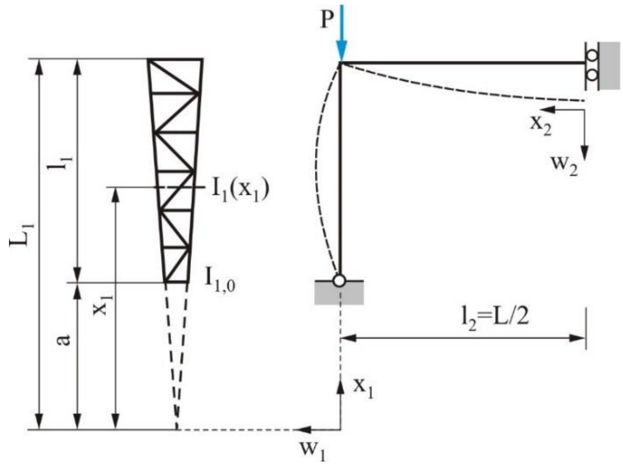
Fig. 2. Possible deformation modes of the rectangular plane frame.

In analysis of the stability of the frame carriers, it is usual to consider each column individually and separated from the frame, i.e. elastically clamped. The column of the variable cross-section, loaded by the compressive axial force *P*, is presented in Fig. 3(a) for the case of the braced or non-sway frame with the fixed nodes while the same column for the case of the unbraced or sway frame with the movable nodes is shown in Figure 3(b).

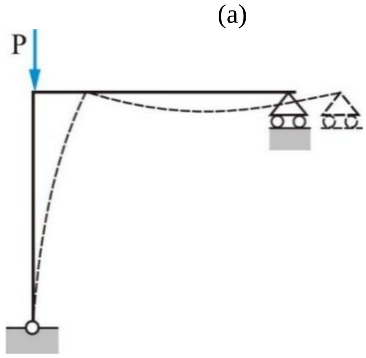
Differential equations, describing the problem of buckling of the perfectly strait column prepared from isotropic homogenous material with variable cross-section loaded by the compressive axial force *P*, for the case of the frame with the fixed nodes (Fig. 3(a)) are:

$$EI_1(x_1)\cdot\frac{d^4w_1}{dx_1^4}+P\cdot\frac{d^2w_1}{dx_1^2}=0,$$
$$\frac{d^4w_2}{dx_2^4}=0, \tag{1}$$

where: $I_1(x_1)= I_{1,0}\cdot(x_1/a)^2$ and $I_{1,0}$ is the moment of inertia at the bottom of the column, E is the Young's elasticity modulus of the column material, w_1 is the lateral deflection of the column, and w_2 is the vertical deflection of the frame's beam. Eventual support settlements and local stability loss of the individual members are not considered. Even the eccentricity influence of connections of individual rods in joints is not taken into account.



(a)



(b)

Fig. 3. Column stability calculation for the frame:
(a) with the fixed and (b) with the movable nodes.

The general solution of the differential equations (1) is [9]:

$$w_1(x_1) = \sqrt{\frac{x_1}{a}} \cdot \left[A_1 \sin\left(\omega \cdot \ln \frac{x_1}{a} \right) + B_1 \cos\left(\omega \cdot \ln \frac{x_1}{a} \right) \right] + C_1 x_1 + D_1$$
$$w_2(x_2) = A_2 x_2^3 + B_2 x_2^2 + C_2 x_2 + D_2. \tag{2}$$

The integration constants are being determined from the boundary conditions:

$$w_1(a) = \frac{d^2 w_1(a)}{dx_1^2} = w_2(\ell_2) = 0, \quad \frac{dw_1(L_1)}{dx_1} = \frac{dw_2(\ell_2)}{dx_2}, \quad w_1(L_1) = 0, \quad w_2'(0) = 0,$$
$$EI_1(L_1) \cdot \frac{d^2 w_1(L_1)}{dx_1^2} + EI_2 \cdot \frac{d^2 w_2(\ell_2)}{dx_2^2} = 0, \quad -EI_2 \cdot \frac{d^3 w_2(0)}{dx_2^3} = 0. \tag{3}$$

With boundary conditions given with (3), one obtains the equation of buckling for the column of the frame with the fixed nodes:

$$\tan\left[\omega\cdot\ln(\ell'_1+1)\right]=\frac{8\cdot\omega\cdot(0.5\cdot\ell'_1+2)^2}{\frac{l_2}{l_1}\cdot\frac{I_m}{I_2}\cdot\ell'_1\cdot(\ell'_1+1)\cdot(4\cdot\omega^2+1)+4\cdot(0.5\cdot\ell'_1+2)^2}, \tag{4}$$

where: I_m is the moment of inertia of the middle cross-section of the column and $\ell'_1=\ell_1/a$, $\omega=\sqrt{\beta^2(0.5+1/\ell'_1)^2-0.25}$ and $\beta=\sqrt{P\ell_1^2/EI_m}$.

Differential equations, describing the problem of buckling of the column with variable cross-section, loaded by the compressive axial force P, for the case of the frame with the movable nodes (Fig. 3(b)) are:

$$\begin{aligned}&-EI_1(L_1)\cdot\frac{d^3w_1(L_1)}{dx_1^3}+P\cdot\frac{dw_1(L_1)}{dx_1}=0\\&w_2(0)=-EI_2\frac{d^2w_2(0)}{dx_2^2}=0.\end{aligned} \tag{5}$$

Using the boundary conditions (3) and eliminating the integration constants, one obtains the equation of buckling for the column of the frame with the movable nodes:

$$\tan\left[\omega\cdot\ln(\ell'_1+1)\right]=\frac{12\cdot\omega\cdot(0.5\cdot\ell'_1+2)^2}{\frac{l_2}{l_1}\cdot\frac{I_m}{I_2}\cdot\ell'_1\cdot(\ell'_1+1)\cdot(4\cdot\omega^2+1)+6\cdot(0.5\cdot\ell'_1+2)^2}. \tag{6}$$

Based on equations (4) and (6), one can obtain the dimensionless variable, the buckling coefficient, relating the applied load with the column properties, for the cases of column of the frames with the fixed and movable nodes as:

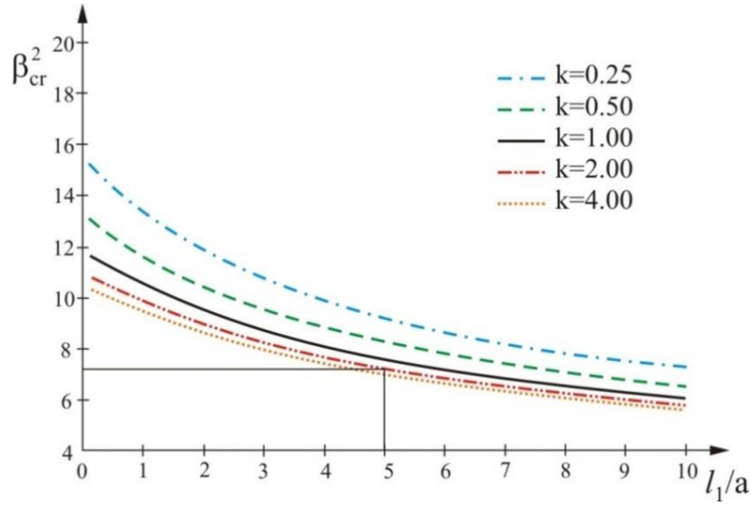
$$\beta_{cr}^2=\frac{P_{cr}\cdot\ell_1^2}{E\cdot I_m}, \tag{7}$$

based on which one can calculate the effective buckling length of the column:

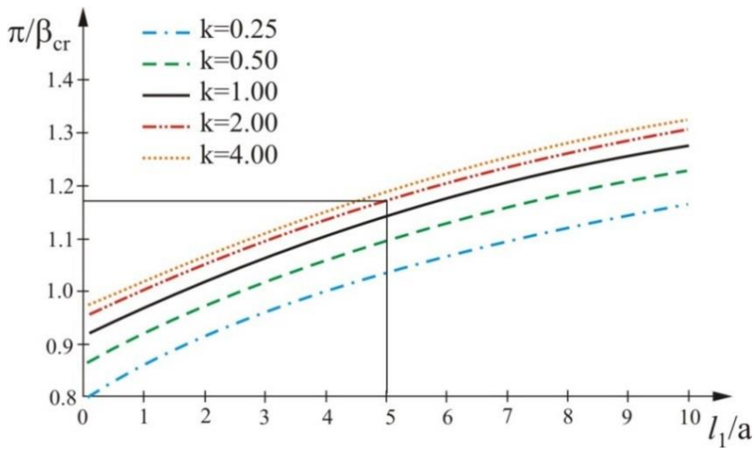
$$\ell_{ef}=\frac{\pi}{\beta_{cr}}\ell_1. \tag{8}$$

3. Results and discussion

The critical load and the effective buckling length are shown in Figs. 4(a) and (b) in terms of the column's buckling length, respectively, for the column of the braced or non-sway frame with the fixed nodes, obtained by the conducted numerical analysis of equation (4) within the program written in the *Mathematica*® environment, for five different values of the $k = (\ell_2 / \ell_1) \cdot (I_m / I_2)$ ratio.



(a)

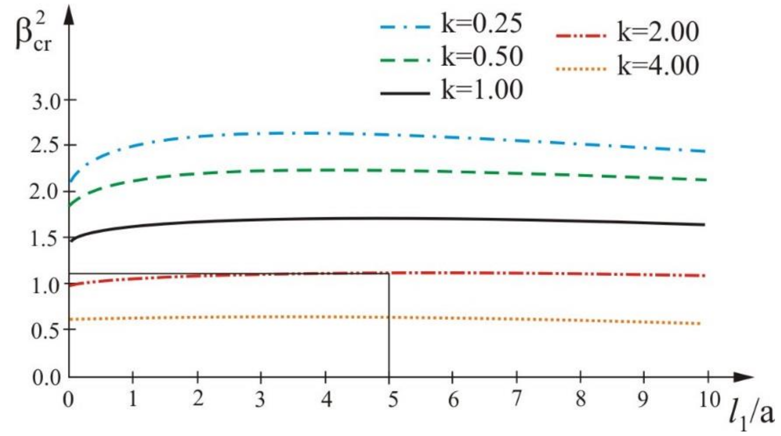


(b)

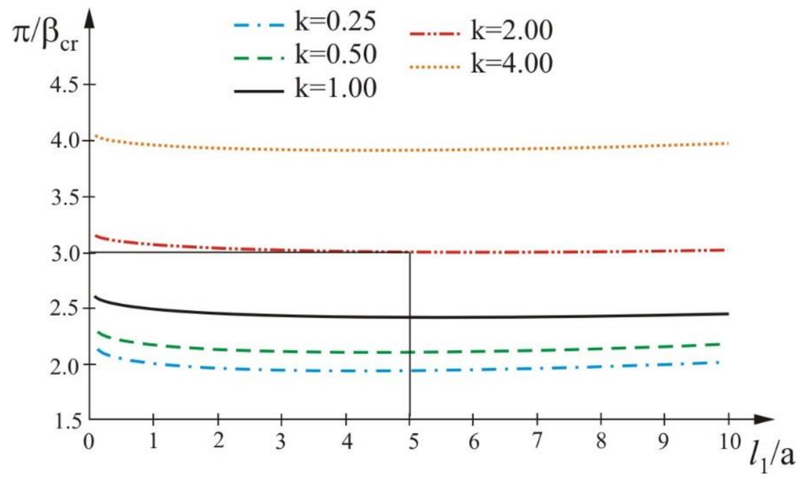
Fig. 4. Diagrams of the critical buckling force (a) and the effective buckling length (b) of the column with the variable cross-section for the case of the braced or non-sway frame with fixed nodes in terms of the column buckling length.

From Fig. 4, one can see that the critical buckling force is decreasing with increase of the column length, while the effective buckling length is increasing.

On the other hand, the critical load and the effective buckling length are shown in Figs. 5(a) and (b) in terms of the columns buckling length, respectively, for the column of the unbraced or sway frame with the movable nodes, obtained by the conducted numerical analysis of equation (6) within the program also written in the *Mathematica*® environment, for five different values of the $k = (\ell_2 / \ell_1) \cdot (I_m / I_2)$ ratio.



(a)



(b)

Fig. 5. Diagrams of the critical buckling force (a) and the effective buckling length (b) of the column with the variable cross-section in terms of the column buckling length, for the case of the unbraced or sway frame with movable nodes.

It can be seen from Fig. 5 that the critical buckling force initially increases with increase of the column length and then mildly decreases, while the effective buckling length initially decreases and then mildly increases.

By comparing results from Figs. 4(a) and 5(a), it can be noticed that the critical buckling force has the smaller values for the column of the *unbraced or sway* frame with the movable nodes, with respect to the *braced or non-sway* frame with the fixed nodes, for the same column length. The effective buckling length is larger for the column of the *unbraced or sway* frame with the movable nodes, for the same column length, what can be seen from comparison of diagrams in Figs. 4(b) and 5(b).

The obtained diagrams can be used for obtaining the critical buckling force and the effective buckling length of the real columns in the following way. Let the column of the variable cross-section have length $\ell_1 = 7.5\ m$, moment of inertia $I_{1,0} = 1\cdot 10^{-4}\ m^4$ and distance $a = 1.5\ m$. Let the beam length be $\ell_2 = 6\ m$ with moment of inertia $I_2 = 5\cdot 10^{-4}\ m^4$. The middle cross-section moment of inertia is $I_m = 12.25\cdot 10^{-4}\ m^4$. The other relevant variables are thus: $(\ell_1/a) = 5$ and $k = 1.96$. From diagram in Fig. 4(a) for values$(\ell_1/a) = 5$ and $k = 2$, one can read off that the value of the critical buckling force is $\beta_{cr}^2 = 7.2$, while the effective buckling length amounts to $\pi/\beta_{cr} = 1.17$, for the column of the frame with the fixed nodes. The corresponding values for the column of the frame with the movable nodes are $\beta_{cr}^2 = 1.18$ and $\pi/\beta_{cr} = 3.0$, respectively.

4. Conclusions

The critical buckling force and the effective buckling length for the columns with the variable cross-section, in the braced or non-sway frames with the fixed nodes and the unbraced or sway frame with the movable nodes, were analyzed in this paper.

By using the differential equilibrium equations and the corresponding boundary conditions, the equations of column buckling are presented; solving of these equations produces the critical buckling force and the effective buckling length of those columns. Values of those variables were obtained for several different parameters (ratio of lengths and heights of frames, ratio of moments of inertia of the cross-sections of the frames parts) for the frames with the fixed and movable nodes.

For the column of the braced or non-sway frame with the fixed nodes, the critical buckling force decreases with increase of the column length, while the effective buckling length increases. For the unbraced or sway frame with the movable nodes, the behavior of these parameters is somewhat different; the critical buckling force initially increases with increase of the column length and then mildly decreases, while the effective buckling length initially decreases and then mildly increases.

From comparison of the investigated parameters for the columns of the frames with the fixed and movable nodes, one can conclude that the critical buckling force has the smaller values for the column of the sway frame, with respect to the braced frame, for the same column length. The effective buckling length is larger for the column of the frame with the movable nodes.

Note: *The shorter version of this research was presented at "XXXXI Active of Experts on Steel Structures", 19.-21. October 2016, Štrbske Pleso, Slovakia, reference [10].*

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Summary

Columns of the variable cross-sections are used in order to decrease the costs of the structure, for the purpose of easier maintenance, or for architectural reasons. Buckling of such columns is a problem that could appear in many steel structures, and could cause the loss of the structural stability. The critical buckling loading and the effective buckling lengths of such columns are analyzed in this paper for the cases of columns in the braced frame with the fixed and the sway frame with the movable nodes. Results are presented on corresponding diagrams what enables pinpointing the exact moment at which the particular column would lose its stability. The comparison of obtained results for the two types of frames are presented, as well.